Journal of Statistical Physics, Vol. 122, No. 2, January 2006 (© 2006)
DOI: 10.1007/s10955-005-8010-1

Triviality of Hierarchical Models with Small Negative ϕ^4 Coupling in Four Dimensions

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Received September 9, 2004; accepted August 1, 2005

The Kadanoff-Wilson renormalization group (RG) for a class of hierarchical spin models including small negative ϕ^4 terms in four dimensions are studied by using Gawędzki and Kupiainen's analysis. We prove triviality for the class, namely prove existence of critical trajectory that leads to the Gaussian fixed point.

KEY WORDS: Hierarchical model; Triviality; Renormalization group; Negative ϕ^4 coupling.

1. INTRODUCTION

Hierarchical spin model is an equilibrium statistical mechanical system introduced by Bleher and Sinai[1,2] as a model suitable for tracing block spin renormalization group (RG) trajectories. For the model, the RG transformation is reduced to the following nonlinear transformation $\mathcal{R}$ of a function (single spin potential) $v = v(\phi)$:

$$\exp[-\mathcal{R}v(\phi)] = \frac{\int \exp\left[-\frac{1}{2}L^d\left[v\left(L^{-(d-2)/2}\phi + z\right) + v\left(L^{-(d-2)/2}\phi - z\right)\right]\right]d\nu(z)}{\int \exp[-L^4 v(z)]d\nu(z)} \tag{1}$$

where $d\nu(z) = \frac{1}{(2\pi)^{1/2}} \exp(-\frac{1}{2}z^2)dz$, and $L \geq 10$ is an even integer valued constant. It is easy to see that the trivial function $v(\phi) \equiv 0$ is a fixed point of $\mathcal{R}$, which we call the Gaussian fixed point. If, for a class of single spin potentials, RG trajectories with initial potentials in the class, converge to the Gaussian fixed

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point, then we say that the class of functions is trivial. Gawędzki and Kupiainen studied this recursion in detail, and proved (among other things) the triviality for ϕ^4 models with some small ϕ^4 coupling constant in 4 dimensions[3–5]. See ref. 5 for a review of their results together with the relation of (1) and the hierarchical spin model.

The purpose of the present paper is to extend the results of Gawędzki and Kupiainen and prove triviality for a wider class of potentials. To be specific, we consider the following class of single spin potentials:

$$v(\phi) = \mu\phi^2 + \left(\lambda - \frac{15\rho}{1 - L^{-2}}\right)\phi^4 + \rho\phi^6, \quad \phi \in \mathbf{R}. \tag{2}$$

Before stating our results, let us briefly review the relative known results. The triviality of the hierarchical model with (2) in 3 dimensions has already been established for small parameters by Müller and Schiemann[8]. A common belief based on power counting type of arguments suggest that the results of ref. 8 would support triviality also for higher dimensions. However, one should note that the role of ϕ^4 term for $d = 3$ is different from that for $d = 4$. It is because ϕ^4 is relevant for $d < 4$ while it is not relevant for $d \geq 4$. The statement in ref. 8 says that for sufficiently small ρ we can find μ and λ such that the RG trajectory converges to the Gaussian fixed point. On the other hand, a triviality statement for $d = 4$ is that for arbitrary (but small) ρ and λ we can find μ such that the same conclusion holds. In the latter case, we have to prove that an arbitrary (small) choices of λ and ρ, in particular, with negative ϕ^4 term, do not distort the standard expectation of the behavior of an RG trajectory. T. Hara, T. Hattori, and H. Watanabe proved the triviality of Dyson's hierarchical Ising model in 4 dimensions. [7] They used characteristic function of single spin distributions and Newman's inequalities on truncated correlations. Their method is useful to analyze in the strong coupling regime. We expect that their method is valid for analyzing general class of initial single spin potentials. However, (2) is still out of the range of their method, because these initial single spin potentials do not satisfy the Lee-Yang property when the coefficient of ϕ^4 term is negative[9], so truncated correlations of these potentials do not satisfy Newman's inequalities that are the key estimation which their method needs[7,10]. Gawędzki and Kupiainen succeeded the construction of the non-trivial Euclidean ϕ_4^4 theory with the complex coupling constant with negative real part[6]. However, the parameter region of the coupling constants which we studied is not included in their studied region.

Let us turn to our proof of triviality for (1) with potentials of the form (2). We will show that the parameters will enter the region where the Theorem of Gawędzki and Kupiainen[5] can be applied (we call this region "G-K region"), after some iterations (finite time iterations) of the RG by the same techniques of Gawędzki and Kupiainen[5]. The point of our proof is to change the

induction hypothesis after some iterations to reflect the dominant terms in the potential.
Now, we state the results precisely. We will use the following notation:

$$v_n(\phi) = \mathcal{R}^n v_0(\phi), \tag{3}$$

$$(v_n)_k(\phi) = \frac{1}{k!}\frac{d^k(v_n)(0)}{d\phi^k}\phi^k, \tag{4}$$

$$(v_n)_{\geq k}(\phi) = v_n(\phi) - \sum_{l<k}(v_n)_l(\phi). \tag{5}$$

Let us be given an initial single spin potential

$$v_0(\phi) = \left(\mu_0 - \frac{1}{2}c_2(v_0)\right)\phi^2 + \left(\lambda_0 - \frac{1}{2}c_4(v_0)\right)\phi^4 + \rho_0\phi^6 + (v_0)_{\geq 8}(\phi), \tag{6}$$

where

$$c_2(v_0) = \frac{12\lambda_0}{1-L^{-2}} - \frac{180\rho_0}{(1-L^{-2})(1-L^{-4})} + \frac{90\rho_0}{1-L^{-4}}, \tag{7}$$

$$c_4(v_0) = \frac{30\rho_0}{1-L^{-2}}, \tag{8}$$

are the counter terms originating from Wick ordering. Let us define a class of initial single spin potentials $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ satisfying the following conditions for constants L, D, C_1, n_0, and ρ_0.

Ta for $|\mathbf{Im}\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ even, and

$$\left|e^{-(v_0)(\phi)}\right| \leq \exp\left[D - \left(\lambda_0^{1/2} + \rho_0^{1/3}\right)|\phi|^2 + A_1\lambda_0(\mathbf{Im}\phi)^4 + A_2\rho_0(\mathbf{Im}\phi)^6\right], \tag{9}$$

where $A_1(\geq 20)$ and $A_2(\geq 2004)$ are universal constants.

Tb for $|\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $(v_0)_{\geq 4}(\phi)$ is analytic,

$$(v_0)_{\geq 4}(\phi) = \left(\lambda_0 - \frac{15\rho_0}{1-L^{-2}}\right)\phi^4 + \rho_0\phi^6 + (v_0)_{\geq 8}(\phi), \tag{10}$$

with

$$\frac{C_{--}L^{-4}}{n_0} \leq \lambda_0 \leq \frac{C_{++}L^{-4}}{n_0}, \quad C_{--} = \frac{1}{42}, \quad C_{++} = \frac{1}{28}, \tag{11}$$

$$|(v_0)_{\geq 8}(\phi)| \leq \rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}. \tag{12}$$

Notice that the coefficient of ϕ^4 is represented as $\lambda_0 - \frac{15\rho_0}{1-L^{-2}}$. If we take a value of ρ_0 suitably large, then the coefficient of ϕ^4 will be negative. So, the class $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ includes small negative ϕ^4 case. This case is the main object of our study in this paper. Of course, this class includes potentials which Gawędzki and Kupiainen studied. In fact it is easy to see that $\mathcal{V}_0(L, D, C_1, n_0, 0)$ is a class which are investigated in.[5] We will prove the following for our class.

Theorem 1.1. *There exist positive constants:*

$$D,\ \bar{C}_1(L, D) \geq L,\ \bar{n}_0(L, D, C_1) \geq L^{48},$$

such that the following holds.

Let $C_1 \geq \bar{C}_1(L, D)$, $n_0 \geq \bar{n}_0(L, D, C_1)$, and

$$0 \leq \rho_0 \leq L^{-4} n_0^{-1}. \tag{13}$$

Define the RG as (1). Then there exists $\mu_{crit} \in \mathbf{R}$ such that the iterates v_n of the recursion converge to zero uniformly on compacts in $\mathbf{C}^1$, if we start from $v_0 \in \mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ with $\mu_0 = \mu_{crit}$.

The proof goes along the following line. In the beginning, we are in the region where $(v_n)_{\geq 6}(\phi)$ is dominant. For properly chosen initial data, $(v_n)_{\geq 6}(\phi)$ decreases rapidly, and we then go into the region where ϕ^4 term of $v_n(\phi)$ is comparable to $(v_n)_{\geq 6}(\phi)$. As the recursion proceeds, the ϕ^4 term becomes positive and dominant, after which it eventually decreases, and $v_n(\phi)$ finally enters the G–K region. To trace the trajectory, we will divide up the induction into 2 parts along the trajectory and impose different induction hypothesis for the ρ dominant regime and the λ dominant regime. (Compare the induction hypotheses L1.2a and L1.2b with L1.3a and L1.3b, respectively.)

We will prove this by means of two lemmas. First, for $n \geq 0$, let $\mathcal{V}_n^1(L, D, C_1, n_0, \rho_0)$ be the class of potentials v_n satisfying:

L1.2a for $|\mathbf{Im}\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $\exp[-v_n(\phi)]$ is analytic, positive for real ϕ, even, and

$$\left|e^{-v_n(\phi)}\right| \leq \exp\left[D - \left(\lambda_n^{1/2} + \rho_n^{1/3}\right)|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4 + A_2\rho_n(\mathbf{Im}\phi)^6\right], \tag{14}$$

L1.2b for $|\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $(v_n)_{\geq 4}(\phi)$ is analytic, and

$$(v_n)_{\geq 4}(\phi) = \left(\lambda_n - \frac{1}{2}c_4(v_n)\right)\phi^4 + \rho_n\phi^6 + (v_n)_{\geq 8}(\phi) \tag{15}$$

with

$$|\lambda_n - \lambda_0| \leq n n_0^{-7/4}, \tag{16}$$

$$|\rho_n - L^{-2n}\rho_0| \leq n L^{-2n} n_0^{-7/4}, \tag{17}$$

$$|(v_n) \geq 8(\phi)| \leq (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}) L^{-n}. \tag{18}$$

Lemma 1.2. *There exist constants D, $\bar{C}_1(L, D) \geq L$, $\bar{n_0}(L, D, C_1) \geq L^{48}$ such that the following holds. Let $C_1 \geq \bar{C}_1(L, D)$, $n_0 \geq \bar{n_0}(L, D, C_1)$ and $n \geq 0$ satisfy the inequality*

$$(n_0 + n)^{1/4} \geq \left(L^{2n-4} \rho_0^{-1}\right)^{1/6}. \tag{19}$$

Suppose also that $v_0(\phi) \in \mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ with

$$L^{-4} n_0^{-3/2} \leq \rho_0 \leq L^{-4} n_0^{-1}, \tag{20}$$

and $v_n(\phi) \in \mathcal{V}_n^1(L, D, C_1, n_0, \rho_0)$.

Then, there exists a closed interval $J_n \subset I_n = [-(n_0 + n)^{-3/2}, (n_0 + n)^{-3/2}]$ such that for μ_n running through J_n, $v_{n+1} \in \mathcal{V}_{n+1}^1(L, D, C_1, n_0, \rho_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

Iterating Lemma 1.2, each time we can choose a subinterval $J_{n+1} \subset J_n \subset I_0$ of the initial mass squared values such that μ_n sweeps I_{n+1}. The effect of relatively large coefficient of ϕ^6 provides us with a positive coefficient of ϕ^4 in the next step, and we can get rid of negative ϕ^4 term. However, we can not iterate Lemma 1.2 for arbitrary times because of assumption (19). In other words, $(v_n)_{\geq 6}(\phi)$ will not be dominant any more compared with ϕ^4 term after some iterations. After applying Lemma 1.2 as much as possible, we are still not in the G-K region, and we have to trace the trajectory carefully for some more steps. So, we must prepare new assumptions. Put

$$n_1 = \max\{n \in \mathbf{N} \big| \left(L^{2n-4} \rho_0^{-1}\right)^{1/6} \leq (n_0 + n)^{1/4}\} + 1. \tag{21}$$

Notice that this number is the first n for which Lemma 1.2 can not be applied. Obviously, we have $n_1 \leq \frac{1}{4} \log_L n_0$.

Let us define a class of single spin potentials $\mathcal{V}_{n_1+n}^2(L, D, C_1, n_0, \rho_0)$ satisfying:

L1.3a for $|\mathbf{Im}\phi| < C_1(n_0 + n_1 + n)^{1/4}$, $\exp[-v_{n_1+n}]$ is analytic and positive for real ϕ, even, and

$$\left|e^{-v_{n_1+n}(\phi)}\right| \leq \exp[D - \left(\lambda_{n_1+n}^{1/2} + \rho_{n_1+n}^{1/3}\right)|\phi|^2 + A_1 \lambda_{n_1+n} (\mathbf{Im}\phi)^4] \times \exp[A_2 \rho_{n_1+n} (\mathbf{Im}\phi)^6], \tag{22}$$

L1.3b for $|\phi| < C_1(n_0 + n_1 + n)^{1/4}$, $(v_{n_1+n})_{\geq 4}(\phi)$ is analytic,

$$(v_{n_1+n})_{\geq 4}(\phi) = \left(\lambda_{n_1+n} - \frac{1}{2} c_4(v_{n_1+n})\right)\phi^4 + \rho_{n_1+n}\phi^6 + (v_{n_1+n})_{\geq 8}(\phi), \tag{23}$$

with

$$|\lambda_{n_1+n} - \lambda_0| \le (n_1 + n) n_0^{-7/4}, \tag{24}$$

$$|\rho_{n_1+n} - L^{-2(n_1+n)}\rho_0| \le (n_1 + n) L^{-2(n_1+n-1)} n_0^{-7/4}, \tag{25}$$

$$|(v_{n_1+n})_{\ge 8}(\phi)| \le L^{-3n-n_1} \left(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}\right). \tag{26}$$

Note that $\mathcal{V}^1_{n_1}(L, D, C_1, n_0, \rho_0) \subset \mathcal{V}^2_{n_1}(L, D, C_1, n_0, \rho_0)$, if L, D, C_1, n_0, and ρ_0 are same constants.

Lemma 1.3. *There exist constants D, $\bar{C}_1(L, D) \ge L$, $\bar{n_0}(L, D, C_1) \ge L^{48}$ such that the following holds.*

Let $C_1 \ge \bar{C}_1(L, D)$, $n_0 \ge \bar{n_0}(L, D, C_1)$, $\log_L n_0 \ge n \ge 0$.

$v_0(\phi) \in \mathcal{V}_0(L, D, C_1, n_0, \rho_0)$, and $v_{n_1+n}(\phi) \in \mathcal{V}^2_{n_1+n}(L, D, C_1, n_0, \rho_0)$.

Then, there exists a closed interval $J_{n_1+n} \subset I_{n_1+n} = [(n_0 + n_1 + n)^{-3/2}, (n_0 + n_1 + n)^{-3/2}]$ such that for μ_{n_1+n} running through J_{n_1+n}, $v_{n_1+n+1} \in \mathcal{V}^2_{n_1+n+1}$. Further, the map $\mu_{n_1+n} \mapsto \mu_{n_1+n+1}$ sweeps I_{n_1+n+1} continuously.

The proof of Lemma 1.3 is close to the proof of Lemma 1.2. A different point from Lemma 1.2 is the difference in the condition of the region where $v_{n_1+n}(\phi)$ satisfies analyticity. In fact we require that $\exp[-v_{n_1+n}(\phi)]$ is analytic for $|\mathbf{Im}\phi| < C_1(n_0 + n_1 + n)^{1/4}$ in Lemma 1.3. The reason why that there is such a difference because ϕ^4 term becomes dominant compared with $(v_{n_1+n})_{\ge 6}(\phi)$. If we notice that the conditions of Lemma 1.3 are different from the conditions of Lemma 1.2, we can prove Lemma 1.3 in a similar way as Lemma 1.2. So, we omit the proof of Lemma 1.3 from this paper.

With Lemma 1.3 we can continue iterations, and we can make sure that after a finite number of iterations, this potential is in the G-K region. More precisely, Gawędzki and Kupiainen introduced a class of potentials $\mathcal{V}^{G-K}_n(L, D, C_1, n_0)$, which is defined to satisfy:

G-Ka $e^{-(v_n)_{\ge 4}(\phi)}$ is analytic in $|\mathbf{Im}\phi| < C_1(n_0 + n)^{1/4}$, positive for real ϕ, even and

$$|\exp[-(v_n)_{\ge 4}(\phi)]| \le \exp[D - \lambda_n^{1/2}|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4], \tag{27}$$

G-Kb for $|\phi| < C_1(n_0 + n)^{1/4}$, $(v_n)_{\ge 4}(\phi)$ is analytic,

$$(v_n)_{\ge 4}(\phi) = \lambda_n\phi^4 + (v_n)_{\ge 6}(\phi) \tag{28}$$

with

$$\frac{C_- L^{-4}}{n_0 + n} \le \lambda_n \le \frac{C_+ L^{-4}}{n_0 + n}, C_- = \frac{1}{48}, C_+ = \frac{1}{24}, \tag{29}$$

$$|(v_n)_{\geq 6}(\phi)| \leq (n_0+n)^{-3/4} \tag{30}$$

In this class $\mathcal{V}_n^{G-K}(L, D, C_1, n_0)$, Gawędzki and Kupiainen proved the following,

Theorem1.4.*(Gawędzki and Kupiainen)* There exist constants D, $\bar{C}_1(L, D)$, $\bar{n}_0$ (L, D, C_1) such that the following holds. Let $C_1 \geq \bar{C}_1(L, D)$, $n_0 \geq \bar{n}_0(L, D, C_1)$ and $n \geq 0$.

Put

$$v_n(\phi) = \mu_n - \frac{6\lambda_n}{1-L^{-2}}\phi^2 + (v_n)_{\geq 4}(\phi) \tag{31}$$

where $(v_n)_{\geq 4}(\phi) \in \mathcal{V}_n^{G-K}(L, D, C_1, n_0)$. Then, there exists a closed interval $J_n \subset I_n$ such that for μ_n running through J_n, $(v_{n+1})_{\geq 4}(\phi) = v_{n+1}(\phi) - \mu_{n+1}\phi^2 + \frac{6\lambda_{n+1}}{1-L^{-2}}\phi^2 \in \mathcal{V}_{n+1}^{G-K}(L, D, C_1, n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

Finally, let us explain differences between the work of Gawędzki and Kupiainen[(5)] and this paper. First of all, we study contribution of the term of ϕ^6 to the term of ϕ^4 rigorously to control the term of ϕ^4 even with negative coefficient. Secondly, our class of single spin potentials permits us to take $v_{\geq 6}(\phi)$ in wider class than that Gawędzki and Kupiainen studied.

2. PROOF OF LEMMA 1.2

We will prove that $v_n'(\phi) = v_{n+1}(\phi)$ is in $\mathcal{V}_{n+1}^1(L, D, C_1, n_0, \rho_0)$, if μ_n is in I_n. We prove this lemma according to Gawędzki and Kupiainen.[(5)] The proof involves the small field region analysis, and the large field region analysis corresponding to the cases either $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$, or $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ respectively.

In the small field, we prove that $v_n'(\phi)$ satisfies **L1.2b**$'$, the condition **L1.2b** with n being replaced by $n+1$, by using the Taylor expansion, and some estimation of the Gaussian integrals as in[(5)].

As for the large field region, we only investigate global behavior of $v_n'(\phi)$, i.e., we confirm that $v_n'(\phi)$ satisfies (18) of **L1.2a**$'$, the condition **L1.2a** with n being replaced by $n+1$.

We use K for calculable absolute constants, whose values will vary in each occurrence.

2.1. Small Field Region Analysis

Let $v_n \in \mathcal{V}_n^1$. Write $\chi_1(z) = \chi(|z| < (L^{2n-4}\rho_0^{-1})^{1/6})$ and throughout this subsection, we assume that ϕ is in the region $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

Note that we have to put C_1 to satisfy the inequality $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$ for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

Next, decompose $v_{n+1}(\phi)$ as follows,

$$v_{n+1}(\phi) = v_n'(\phi) = \overset{\sim}{v_n'}(\phi) + \overset{\approx}{v_n'}(\phi), \tag{32}$$

$$e^{-\overset{\sim}{v_n'}(\phi)} = \int \exp\left[-\frac{L^4}{2}\sum_{\pm} v_n(L^{-1}\phi \pm z)\right] d\nu_1(z)/(\phi = 0)_{\text{small}}, \tag{33}$$

where

$$(\phi = 0)_{\text{small}} = \int \exp[-L^4 v_n(z)] d\nu_1(z), \tag{34}$$

$$d\nu_1(z) \equiv \chi_1(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}. \tag{35}$$

First of all, we estimate $\overset{\sim}{v_n'}(\phi)$ in 2.1.1, and then complete the analysis in the small field region by looking into the Taylor coefficients in 2.1.2

2.1.1. Estimation of $\overset{\sim}{v_n'}(\phi)$

Let us take a logarithm of (33).

$$\begin{aligned}\overset{\sim}{v_n'}(\phi) = {} & L^2\left(\mu_n - \frac{1}{2}c_2(v_n)\right)\phi^2 + \left(\lambda_n - \frac{1}{2}c_4(v_n)\right)\phi^4 + L^{-2}\rho_n\phi^6 \\ & - \log\int e^{-w_\phi(z)}d\nu_1(z) + \log(\phi = 0)_{\text{small}}.\end{aligned} \tag{36}$$

$$w_\phi(z) = w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), \tag{37}$$

and

$$\begin{aligned} w_0(z) &= L^4 v_n(z) \\ &= L^4\left\{\left(\mu_n - \frac{1}{2}c_2(v_n)\right)z^2 + \left(\lambda_n - \frac{1}{2}c_4(v_n)\right)z^4 + \rho_n z^6 + (v_n)_{\geq 8}(z)\right\}, \\ w_2(z) &= L^2\left(6\left(\lambda_n - \frac{1}{2}c_4(v_n)\right)z^2 + 15\rho_n z^4 + \frac{d^2}{2dz^2}(v_n)_{\geq 8}(z)\right), \end{aligned} \tag{38}$$

$$w_4(z) = 15\rho_n z^2 + \frac{1}{4!}\frac{d^4}{dz^4}(v_n)_{\geq 8}(z), \tag{39}$$

$$w_6(z) = L^{-2}\frac{d^6}{6!dz^6}(v_n)_{\geq 8}(z), \tag{40}$$

$$w_{\geq 8}(\phi, z) = \frac{L^{-4}\phi^8}{7!}\left\{\int_0^1 dt(1-t)^7\frac{d^8}{dz^8}(v_n)_{\geq 8}(L^{-1}t\phi + z)\right.$$
$$\left. + \int_0^1 dt(1-t)^7\frac{d^8}{dz^8}(v_n)_{\geq 8}(L^{-1}t\phi - z)\right\}. \tag{41}$$

From the conditions **L1.2a**–**L1.2b**, $v_n(\phi)$ is even and analytic.

We can estimate $\frac{d^8}{dz^8}(v_n)_{\geq 8}(\phi)$ on the support of $d\nu_1(z)$ as follows by using the Cauchy formula and (18),

$$|(v_n)_{\geq 8}(z)| \leq \frac{1}{7!}\int_0^1 dt(1-t)^7|z^8\frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)|$$
$$\leq \frac{C_1}{8!(C_1-1)^9}\left(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}\right)(L^4\rho_0)^{4/3}L^{-11n/3}z^8. \tag{42}$$

$\frac{d^2}{dz^2}(v_n)_{\geq 8}(z)$ to $\frac{d^6}{dz^6}(v_n)_{\geq 8}(z)$ can be estimated as (42).

From the perturbation expansion:

$$-\log\int e^{-w_\phi(z)}d\nu_1(z)$$
$$= -\log\int d\nu_1(z) + \langle w_\phi(z)\rangle_0 - \int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t, \tag{43}$$

where

$$\langle\cdots\rangle_t \equiv \int\cdots e^{-tw_\phi(z)}d\nu_1(z)/\int e^{-tw_\phi(z)}d\nu_1(z). \tag{44}$$

Now, we shall estimate each part of (43). Using the estimation of the Gaussian integrations, we get

$$\langle w_\phi(z)\rangle_0 = L^4\langle v_n(z)\rangle_0 + 6L^2\left(\lambda_n - \frac{1}{2}c_4(v_n)\right)\phi^2 + 45L^2\rho_n\phi^2$$
$$+ \overset{\sim 0,0}{R_2}(L, n_0, \rho_0, n)\phi^2 + 15\rho_n\phi^4 + \overset{\sim 0,0}{R_4}(L, n_0, \rho_0, n)\phi^4$$
$$+ \overset{\sim 0,0}{R_6}(L, n_0, \rho_0, n)\phi^6 + \langle w_{\geq 8}(\phi, z)\rangle_0, \tag{45}$$

where, the terms $\overset{\sim 0,0}{R_{2i}}(L, n_0, \rho_0, n)$, $i = 1, \ldots, 3$ satisfy

$$\left|\overset{\sim 0,0}{R_{2i}}(L, n_0, \rho_0, n)\right| \leq \left(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}\right)(L^4\rho_0)^{4/3}L^{-11n/3}. \tag{46}$$

Therefore from (45), we can estimate $\langle w_{\geq 8}(\phi, z)\rangle_0$ as follows,

$$|\langle w_{\geq 8}(\phi, z)\rangle_0| \leq L^{4-n}(1 + (L^4\rho_0)^{1/3} L^{-2n/3})\big(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}\big). \tag{47}$$

Next we estimate

$$\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t = \int_0^1 dt(1-t)\sum_{i,j}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t$$

$$= \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t + \int_0^1 dt(1-t)\sum_{i,j\neq 0}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t, \tag{48}$$

where

$$\tilde{w}_{2i} = \begin{cases} w_{2i}(z)\phi^{2i} & i = 0, 1, 2, 3 \\ w_{\geq 8}(\phi, z) & i = 4. \end{cases}$$

The cumulants are

$$\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t = \langle e^{-tw_\phi(z)}\rangle_0^{-1}\langle \tilde{w}_{2i}\tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0 - \langle e^{-tw_\phi(z)}\rangle_0^{-2}\langle \tilde{w}_{2i}e^{-tw_\phi(z)}\rangle_0\langle \tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0. \tag{49}$$

Note that the support of $d\nu_1(z)$ is $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and the coefficients are as small as the inverse of the diameter of the small field region.

We get the uniform estimate $|w_\phi(z)| \leq K \cdot C_1^4$ for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < \frac{10LC_1}{11}(L^{2n-4}\rho_0^{-1})^{1/6}$. We have

$$\left|\sum_{(i,j)\neq(0,0)}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t\right| \leq e^{K\cdot C_1^4}\sum_{(i,j)\neq(0,0)}\big(\langle|\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0 + \langle|\tilde{w}_{2i}|\rangle_0\langle|\tilde{w}_{2j}|\rangle_0\big). \tag{50}$$

So, we can estimate $|\int_0^1 dt(1-t)\sum_{(i,j)\neq(0,0)}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t|$ similarly as in (42), we obtain

$$|\text{2nd term of RHS of (48)}| \leq Ke^{K\cdot C_1^4}L^{-2}n_0^{-2}(|\phi|^2 + L^{-2}|\phi|^4 + L^{-2n-4}|\phi|^6) + |\text{higher order terms}|. \tag{51}$$

The higher order terms are estimated as follows,

$$|\text{higher order terms}| \leq Ke^{K\cdot C_1^4}L^{12-n}C_1^{12}n_0^{-1/8}\big(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}\big). \tag{52}$$

Next, we estimate $\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t$. From the Taylor expansion of ϕ, and we can use the Cauchy formula because $\langle w_0(z); w_0(z)\rangle_t$ is analytic function in $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, we get

$$\left|\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}\right|$$

$$\leq K \exp(K \cdot C_1^4) \cdot L^{-2} n_0^{-2} |\phi|^2. \tag{53}$$

So we have,

$$\left| \int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0} \right|$$
$$\leq K \exp\left(K \cdot C_1^4\right) L^{-2} n_0^{-2} (|\phi|^2 + L^{-2}|\phi|^4 + L^{-2n-4}|\phi|^6)$$
$$+|\text{higher order terms}|, \tag{54}$$

$$|\text{higher order terms}| \leq K e^{K \cdot C_1^4} L^{12-n} C_1^{12} n_0^{-1/8} \left(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}\right). \tag{55}$$

These coefficients are large, but not terrible, because we can take n_0 sufficiently large. In the following, we put $n_0^{1/8} \geq K \cdot C_1^{12} L^{12} e^{K \cdot C_1^4}$.

From (36) and (43), we infer that

$$\tilde{v}_n' (\phi) = L^2\left(\mu_n - \frac{1}{2}c_2(v_n)\right)\phi^2 + 6L^2 \left(\lambda_n - \frac{1}{2}c_4(v_n)\right)\phi^2 + 45L^2 \rho_n \phi^2$$
$$+ \tilde{R}_2 (L, n_0, \rho_0, n)\phi^2 + (\lambda_n - \frac{1}{2}c_4(v_n))\phi^4 + 15\rho_n\phi^4 + \tilde{R}_4 (L, n_0, \rho_0, n)\phi^4$$
$$+L^{-2}\rho_n\phi^6 + \tilde{R}_6 (L, n_0, \rho_0, n)\phi^6 + (\tilde{v_n})'_{\geq 8} (\phi), \tag{56}$$

where, the terms $\tilde{R}_{2i} (L, n_0, \rho_0, n)$, $i = 1, 2, 3$ satisfy

$$|\tilde{R}_{2i}(L, n_0, \rho_0, n)| \leq L^{-10-2i} n_0^{-2+1/8} + \left| \tilde{R}_{2i}^{0,0} (L, n_0, \rho_0, n)\right|, i = 1, 2, \tag{57}$$

$$\left|\tilde{R}_6(L, n_0, \rho_0, n)\right| \leq L^{-2n-18} n_0^{-2+1/8} + \left|\tilde{R}_6^{0,0}(L, n_0, \rho_0, n)\right|, \tag{58}$$

and from (47) and (55), $(\tilde{v_n})'_{\geq 8} (\phi)$ satisfy

$$|(\tilde{v_n})'_{\geq 8} (\phi)| \leq L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4})\left(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}\right), \tag{59}$$

for $|\phi| < \frac{10}{11} L C_1(\rho_0^{-1} L^{2n})^{1/6}$. Notice that

$$(\phi = 0)_{\text{small}} = \log \int d\nu_1(z) - \langle w_0(z)\rangle_0 + \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}.$$

So we can check that the constant term $(\phi = 0)_{\text{small}}$ vanishes. The estimate (59) is a little weaker than what we want (see (18)). So, we need a stronger estimate. Since $\tilde{v}_n' (\phi)$ is analytic in $|\phi| < \frac{10}{11} L C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $\phi^{-8} (\tilde{v_n})'_{\geq 8} (\phi)$ is also analytic in $|\phi| < \frac{10}{11} L C_1(L^{2n-4}\rho_0^{-1})^{1/6}$. We obtain from the maximum principle

$$|(\tilde{v_n})'_{\geq 8}(\phi)| \leq \left(\frac{|\phi|}{(10L/11)C_1(L^{2n-4}\rho_0^{-1})^{1/6}}\right)^8 (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4})$$

$$\times(L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4})), \tag{60}$$

so that for $|\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}$,

$$\begin{aligned} |\,\widetilde{(v_n)}'_{\geq 8}\,(\phi)| \leq \left(\frac{11}{10}\right)^8 L^{-16/3}\big(L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4}) \\ \times\big(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}\big)\big). \end{aligned} \tag{61}$$

2.1.2. *Estimation of* $\overset{\approx}{v}{}'_n\,(\phi)$ *for* $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$

Represent (32) as

$$\begin{aligned} \overset{\approx}{v}{}'_n\,(\phi) = \log\left(1 + \frac{\int \exp\big[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)\big](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi = 0)_{\text{small}}}\right) \\ + \log(\phi = 0)_{\text{small}} - \log(\phi = 0). \end{aligned} \tag{62}$$

We want to prove that $\overset{\approx}{v}{}'_n\,(\phi)$ is analytic in$|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ and sufficiently smaller than $\tilde{v}'_n\,(\phi)$. To prove these properties, we have only to prove that

$$\frac{\int \exp\big[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)\big](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi = 0)_{\text{small}}} \tag{63}$$

is analytic and sufficiently small in $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

First of all, we estimate the denominator of (63). We can show that the denominator is bounded from below by a constant which depends on C_1, but not on n_0. From **L1.2b**, and (57–58) together with uniform estimate of $w_0(z)$ under the condition that $(n_0 + n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, we estimate denominator as follows,

$$|\text{denominator of (63)}| \geq \exp\big[-K \cdot C_1^6\big]. \tag{64}$$

Next, we estimate the numerator part of (63),

$$|\text{numerator of (63)}| \leq \int (1-\chi_1(z)) \prod_{\pm} |\exp[-v_n(L^{-1}\phi \pm z)]|^{L^4/2} d\nu(z). \tag{65}$$

Using (14) of **L1.2a** for $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0)^{1/6}$, we have

$$|\text{numerator of (63)}| \leq \exp[K + L^4 D + A_1 C_1^4 + 2A_2 C_1^6 - \frac{1}{4}(L^{2n-4}\rho_0^{-1})^{1/3}]. \tag{66}$$

So,

$$|(63)| < \exp\left[K \cdot C_1^6 + L^4 D + A_1 C_1^4 + 2A_2 C_1^6 - \frac{1}{4}\left(L^{2n-4}\rho_0^{-1}\right)^{1/3}\right]. \quad (67)$$

For given L, D and C_1, we can take n_0 large enough to obtain

$$\text{RHS of (67)} \leq \exp\left[-\frac{1}{8}(L^{2n-4}\rho_0^{-1})^{1/3}\right]. \quad (68)$$

This estimate is also valid for $\log(\phi = 0) - \log(\phi = 0)_{\text{small}}$. According to (68), we can show that $\overset{\approx}{v'_n}(\phi)$ is analytic and

$$|\overset{\approx}{v'_n}(\phi)| \leq 2e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}. \quad (69)$$

2.1.3. Estimation of coefficients

Now, we assume that $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ i.e. ϕ is in the small field region of $v'_n(\phi)$. Notice that the small field region is in the region $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, so we can use the result in 2.1.2 Thus, $\overset{\approx}{v'_n}(\phi)$ is analytic in the small field region, and we can obtain power series expansion of $\overset{\approx}{v'_n}(\phi)$. With the use of Cauchy's estimate, we see that coefficients of ϕ^2, ϕ^4 and ϕ^6 satisfy,

$$\left|\frac{1}{k!}\frac{d^k}{d\phi^k}\overset{\approx}{v'_n}(0)\right| \leq e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}, \quad k = 2, 4, \text{and } 6. \quad (70)$$

Using the bounded convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_n}(0)$, $\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_n}(0)$, and $\frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_n}(0)$ are continuous functions of μ_n on I_n. From (61) and (69), if n_0 is sufficiently large, then we have

$$|(v_n)'_{\geq 8}(\phi)| \leq L^{-(n+1)}\left(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}\right), \quad (71)$$

for $|\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}$. From (56), (58), and (70), we know that

$$|\rho'_n - L^{-2}\rho_n| = |R_6(L, n_0, \rho_0, n) + \frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_n}(0)| \leq L^{-2n}n_0^{-15/8}. \quad (72)$$

Thus, if n_0 is sufficiently large, we have

$$\left|\rho'_n - L^{-2(n+1)}\rho_0\right| < (n+1)L^{-2n}n_0^{-7/4} \quad (73)$$

which proves (18) of $\mathbf{L1.2b'}$.

From (56), (57), we know

$$|\lambda_n' - \lambda_n\Big| = |R_4(L, n_0, \rho_0, n) + \frac{d^4}{4!d\phi^4}\overset{\approx}{v_n'}(0) + \frac{15(\rho_n' - L^{-2}\rho_n)}{1-L^{-2}}\Big|$$
$$\leq n_0^{-15/8}. \tag{74}$$

Thus, we have

$$|\lambda_n' - \lambda_0| < (n+1)n_0^{-7/4}, \tag{75}$$

which completes the proof of **L1.2b**$'$. Similarly, we get estimation of coefficient μ_n' as follows,

$$|\mu_n' - L^2\mu_n| \leq \Big|\Big(\frac{6(\lambda_n - \lambda_n')}{1-L^{-2}} - \frac{90(L^{-2}\rho_n - \rho_n')}{(1-L^{-2})(1-L^{-4})} + \frac{45(L^{-2}\rho_n - \rho_n')}{1-L^{-4}}\Big)\Big|$$
$$+|R_2(L, n_0, \rho_0, n) + \frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v_n'}(0)| \leq K \times n_0^{-15/8}. \tag{76}$$

We know that map $R : \mu \mapsto \mu'$ is continuous, and image $R(I_n)$ include I_{n+1}. So that we can take for J_{n+1} a connected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$.

This ends the analysis of the small field properties.

2.2. Large Field Region Analysis

Next, we prove that $e^{-(v_n)'(\phi)}$ satisfy the condition **L1.2a**$'$. First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$. Next, we prove it in $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ i.e. this region includes the small field region of $v'(\phi)$.

2.2.1. The Case Where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$

Note that the definition of the RG (1) has the following expression

$$e^{-v_n'(\phi)} = \int\prod_{\pm}\exp[-v_n(L^{-1}\phi \pm z)]^{L^4/2}d\nu(z)/(\phi=0). \tag{77}$$

$|\mathbf{Im}(L^{-1}\phi \pm z)| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, if $|\mathbf{Im}\phi| < C_1(L^{2n-2}\rho_0^{-1})^{1/6}$. From the condition **L1.2a**,

$$\left|e^{-(v_n)'(\phi)}\right| \leq \exp\left[L^4 D - L^2\left(\lambda_n^{1/2} + \rho_n^{1/3}\right)|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4\right]$$
$$\times \exp[A_2 L^{-2}\rho_n(\mathbf{Im}\phi)^6]\int_{-\infty}^{\infty} e^{-L^4(\lambda_n^{1/2}-\rho_n^{1/3})z^2}d\nu(z)/(\phi=0). \tag{78}$$

Note that, λ_n and ρ_n are positive and sufficiently small, hence, this integral part and $(\phi = 0)$ estimated as absolute constants, so we get

$$\text{RHS of (78)} \leq \exp\left[L^4 D - L^2\left(\lambda_n^{1/2} + \rho_n^{1/3}\right)|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4\right]$$
$$\times \exp[A_2 L^{-2}\rho_n(\mathbf{Im}\phi)^6 + K]. \tag{79}$$

If D and L are given, we take C_1 sufficiently large and then we take n_0 sufficiently large. Thus, we obtain

$$|\exp(-v'(\phi))|$$
$$< \exp[D - \left(\lambda_n'^{\,1/2} + \rho_n'^{\,1/3}\right)|\phi|^2 + A_1\lambda_n'(\mathbf{Im}\phi)^4 + A_2\rho_n'(\mathbf{Im}\phi)^6], \tag{80}$$

for $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$, $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$.

2.2.2. The Case Where $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0)^{1/6}$

Let $(n_0 + n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, $\mu_n \in I_n$, and $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. From (59), (73), (75), and (76), we have

$$\left|e^{-(v_n)'(\phi)}\right| \leq \exp\left[K \cdot L^{-2}C_1^2 n_0^{-1/2}\right]$$
$$\times \exp\left[+K \cdot C_1^4 L^{4/3}\left(L^{-2(n+1)}\rho_0\right)^{1/3}\right]$$
$$\times \exp\left[-\lambda_n'(\mathbf{Re}\phi^4) - \rho_n'(\mathbf{Re}\phi^6) + L^4 n_0^{-1/2}\right]. \tag{81}$$

And, we estimate $\rho_n'(\mathbf{Re}\phi^6)$ as follows,

$$\rho_n'(\mathbf{Re}\phi^6) \geq \rho_n'\left(\frac{1}{4}(\mathbf{Re}\phi)^6 - 2001(\mathbf{Im}\phi)^6\right)$$
$$\geq -\frac{1}{2}D_6 + 2(\rho_n')^{1/3}|\phi|^2 - A_2\rho_n'(\mathbf{Im}\phi)^6. \tag{82}$$

Notice that D_6 does not depend on C_1, n_0 or n. Similarly, we can estimate, $\lambda_n'\mathbf{Re}\phi^4 \geq -\frac{1}{2}D_4 + 2(\lambda_n')^{1/2}|\phi|^2 - A_1\lambda_n'(\mathbf{Im}\phi)^4$. Notice that D_4 does not depend on C_1, n_0 or n, either. Put $D = D_4 + D_6$. From (81) to (82),

$$\left|e^{-(v_n)'(\phi)}\right| \leq \exp[D - ((\lambda_n')^{1/2} + (\rho_n')^{1/3})|\phi|^2 + A_1\lambda_n'(\mathbf{Im}\phi)^4]$$
$$\times \exp\left[+A_2\rho_n'(\mathbf{Im}\phi)^6 - \frac{1}{2}D + K \cdot L^{-2}C_1^2 n_0^{-1/2}\right]$$
$$\times \exp\left[K \cdot L^{4/3}C_1^4\left(L^{-2(n+1)}\rho_0\right)^{1/3} + L^4 n_0^{-1/2}\right], \tag{83}$$

which is smaller than

$$\exp[D - ((\lambda_n')^{1/2} + (\rho_n')^{1/3})|\phi|^2 + A_1\lambda_n'(\mathbf{Im}\phi)^4 + A_2\rho_n'(\mathbf{Im}\phi)^6], \tag{84}$$

if n_0 is sufficiently large. Proof of Lemma 1.2 is completed.

3. PROOF OF THEOREM 1.1

Finally, we prove Theorem 1.1, using Lemma 1.2, Lemma 1.3 and Theorem 1. First of all, we notice that it is possible to take constants L, D, $C_1(L, D)$, $n_0(L, D, C_1)$ to satisfy Lemma 1.2, Lemma 1.3, and Theorem 1. We can check that potential $v(\phi)$ can be iterated n_1 times if initial parameters satisfy the conditions **Ta** and **Tb** because of Lemma 1.2. Notice that $v_{n_1}(\phi)$, the potential after n_1 iterations, satisfies the conditions **L1.3a** and **L1.3b** with $n = 0$, and so Lemma 1.3 can be applied to this potential. We have to iterate $\mathcal{R}$ using Lemma 1.3, sufficiently many times so that the iterated potentials satisfy the G-K conditions. Put

$$n_2 = \min\{n \in \mathbf{N} : |\rho_{n_1+n}||\phi|^6 + |(v_{n_1+n})_{\geq 8}(\phi)| < (n_0 + n_1 + n)^{-3/4} \quad \text{for } |\phi| < C_1(n_0 + n_1 + n)^{1/4}\} + 1. \tag{85}$$

Then,

$$\rho_{n_1+n_2-1} < (n_0 + n_1 + n_2 - 1)^{-9/4}. \tag{86}$$

By calculation, n_2 can be estimated as $n_2 < \log_L n_0$. Since, $\rho_{n_1+n_2} \geq 0$, and by (24)

$$\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1 - L^{-2}} < \lambda_0 + (n_1 + n_2)n_0^{-7/4} < \frac{C_{++}}{L^4} n_0^{-1} + 2(\log_L n_0) n_0^{-7/4} < \frac{C_+}{L^4}(n_0 + n_1 + n_2)^{-1}. \tag{87}$$

Similarly, by (86) we have

$$\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1 - L^{-2}} > \frac{C_-}{L^4}(n_0 + n_1 + n_2)^{-1}. \tag{88}$$

So, we checked the condition **G-Kb** completely.

Next, let us check the condition **G-Ka**. Notice that analyticity, positivity for real ϕ, and even function of $v_{n_1+n_2}(\phi)$ are checked easily. Now, We check the bound of $v_{n_1+n_2}(\phi)$

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp\left[D - \left(\lambda_{n_1+n_2}^{1/2} + \rho_{n_1+n_2}^{1/3}\right)|\phi|^2\right] \times \exp[+A_1\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4 + A_2\rho_{n_1+n_2}(\mathbf{Im}\phi)^6]. \tag{89}$$

From the definitions of n_1 and n_2, $-\rho_{n_1+n_2}^{1/3}|\phi|^2 + A_2\rho_{n_1+n_2}(\mathbf{Im}\phi)^6$ is nonpositive for $(\mathbf{Im}\phi) < C_1(n_0 + n_1 + n_2)^{1/4}$, so we have

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp\left[D - \lambda_{n_1+n_2}^{1/2}|\phi|^2 + A_1\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4\right]. \tag{90}$$

We have checked all of the G-K conditions. Since $\rho_{n_1+n_2-1}$ is sufficiently small by (86), we know

$$\left|\mu_{n_1+n_2} - L^2\left(\mu_{n_1+n_2-1} + \frac{90\rho_{n_1+n_2-1}}{(1-L^{-2})(1-L^{-4})} - \frac{45\rho_{n_1+n_2-1}}{1-L^{-4}}\right)\right|$$

$$\leq 16 n_0^{-15/8} + \left|\frac{15L^2\rho_{n_1+n_2-1}}{1-L^{-2}}\right| \leq K \cdot n_0^{-15/8}. \tag{91}$$

As in the proof Lemmas 1.2 and 1.3, we can take for $J_{n_1+n_2}$ a suitable connected component. So, we can adapt Theorem 1.4. Theorem 1.1 is now proved.

ACKNOWLEDGMENTS

The author is grateful to Professor Tetsuya Hattori for emphasizing the importance of renormalization group idea, and Professor Yasunari Higuchi for his encouragements and his helpful comments and Professor Takashi Hara, and Professor Hiroshi Watanabe for their valuable advices, and honorable referees for their many kind and helpful suggestions.

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